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Sequence with K_1, K_2, K_n, K_{n+1} Mutually Tangent Circles

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Abstract. In this article is given the formula for radius of circle K_n, where in sequence $\{K_j\}$, four circles K_1, K_2, K_n, K_{n+1}, for all $n \geq 3$, are mutually tangent. Radius r_n is expressed in terms of radii r_1, r_2, r_3.

Four circles K_1, K_2, K_3, K_4, with centers and radii O_j, r_j $(j = 1, 2, 3, 4)$ are mutually tangent what means that each of them is tangent to other three. From Descartes-Soddy formula

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}} \tag{1}$$

we get $r_4 = f(r_1, r_2, r_3)$. If we, as in Fig. 1, inscribe circles $K_5, K_6, \ldots, K_{n-1}, K_n$ then we have $r_k = f(r_1, r_2, r_{k-1})$ for all $k = 4, \ldots, n$.

The following problem appeares: Is it possible to express r_n in closed form as a function of first radii r_1, r_2, r_3?

Theorem 1.

$$\begin{aligned} \frac{1}{r_n} &= (n-3)^2 \left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{1}{r_3} + \\ &\quad + 2(n-3)\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}, \qquad (n \geq 4). \end{aligned} \tag{2}$$

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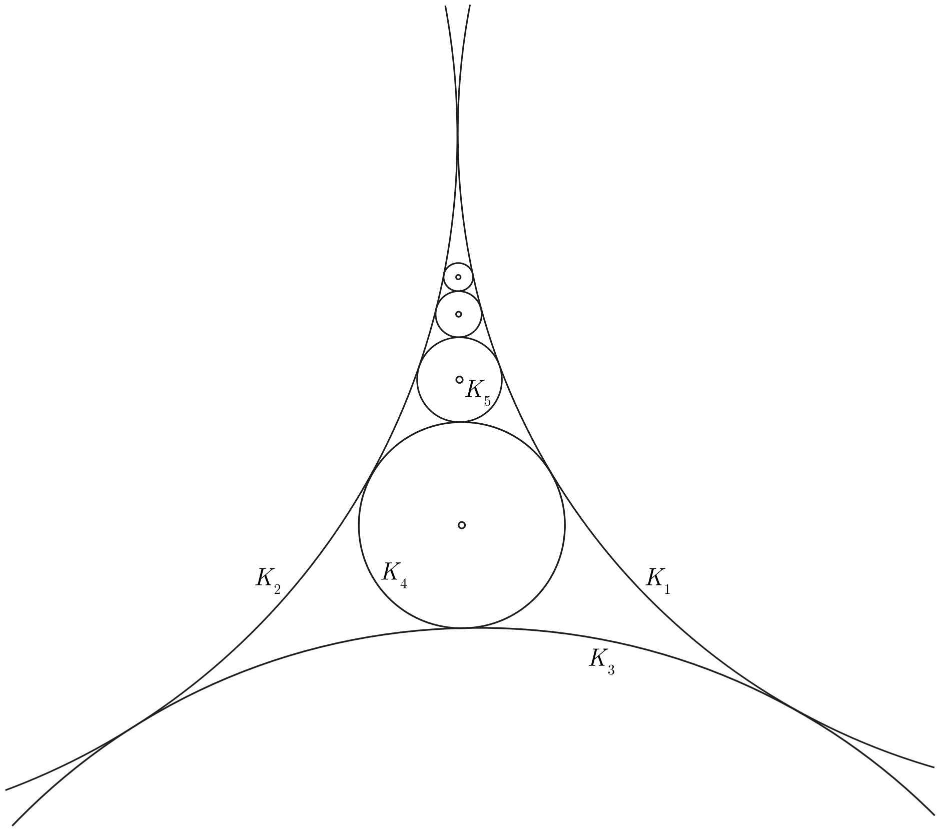

Fig. 1.

Proof. Formula (2) can be proved by induction. For $n = 4$ we have Descartes-Soddy formula. Also, we have

$$\begin{aligned}\frac{1}{r_{k+1}} &= \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_k} + 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_k} + \frac{1}{r_k r_1}} = \\ &= \frac{1}{r_1} + \frac{1}{r_2} + (k-3)^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{1}{r_3} + \\ &\quad + 2(k-3)\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}} + 2\sqrt{\frac{1}{r_1 r_2} + \left(\frac{1}{r_1} + \frac{1}{r_2}\right) M} = \\ &= [(k-3)^2 + 1]\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{1}{r_3} + 2(k-3)\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}} + \\ &\quad + 2\left[(k-3)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}\right] = \\ &= (k-2)^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{1}{r_3} + 2(k-2)\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}},\end{aligned}$$

where

$$M = (k-3)^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+\frac{1}{r_3}+2(k-3)\sqrt{\frac{1}{r_1r_2}+\frac{1}{r_2r_3}+\frac{1}{r_3r_1}},$$

which proves formula (2). □

Formula (2) can be used in finding radii r_n in various configurations. Some Arbelos configurations of inscribed circles will be considered.

Case 1. (Fig. 2):

$$r_1+r_2=r_0, \qquad \frac{1}{r_3}+\frac{1}{r_0}=\frac{1}{r_1}+\frac{1}{r_2},$$

$$\sqrt{\frac{1}{r_1r_2}+\frac{1}{r_2r_3}+\frac{1}{r_3r_1}}=\frac{1}{r_1}+\frac{1}{r_2},$$

$$\frac{1}{r_n}=(n-2)^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)-\frac{1}{r_0}.$$

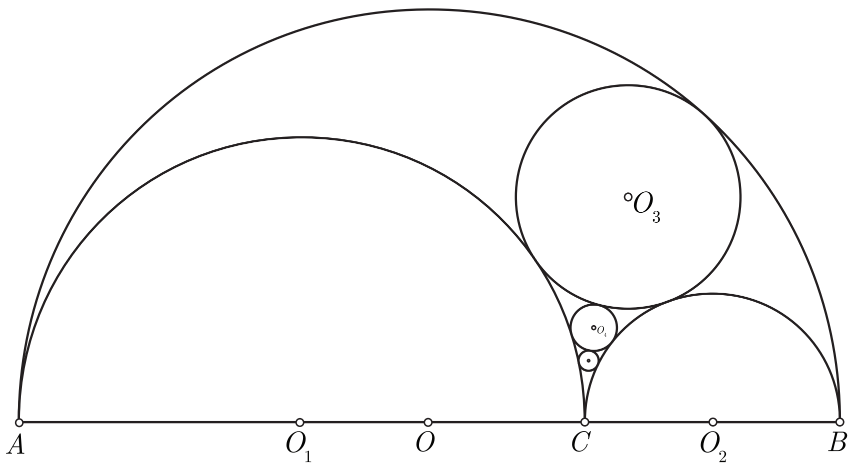

Fig. 2.

Case 2. (Fig. 3):

$$r_1+r_2=r_0, \qquad \frac{1}{r_3}+\frac{1}{r_0}=\frac{1}{r_1}+\frac{1}{r_2},$$

$$\sqrt{\frac{1}{r_1r_3}-\frac{1}{r_3r_0}-\frac{1}{r_0r_1}}=\frac{1}{r_1}-\frac{1}{r_0},$$

$$\frac{1}{r_{k+1}}=\frac{1}{r_1}-\frac{1}{r_0}+\frac{1}{r_k}+2\sqrt{\frac{1}{r_1r_k}-\frac{1}{r_kr_0}-\frac{1}{r_0r_1}}, \qquad (k\geq 3).$$

Formula similar to formula (2) with $\frac{1}{r_2}\to-\frac{1}{r_0}$ in this case is

$$(2') \qquad \frac{1}{r_n}=(n-3)^2\left(\frac{1}{r_1}-\frac{1}{r_0}\right)+\frac{1}{r_3}+2(n-3)\sqrt{\frac{1}{r_1r_3}-\frac{1}{r_3r_0}-\frac{1}{r_0r_1}}.$$

which leads to the formula for radius of n-th left circle

$$\frac{1}{r_n} = (n-2)^2\left(\frac{1}{r_1} - \frac{1}{r_0}\right) + \frac{1}{r_2},$$

and formula for radius of n-th right circle is

$$\frac{1}{r_n} = (n-2)^2\left(\frac{1}{r_2} - \frac{1}{r_0}\right) + \frac{1}{r_1}.$$

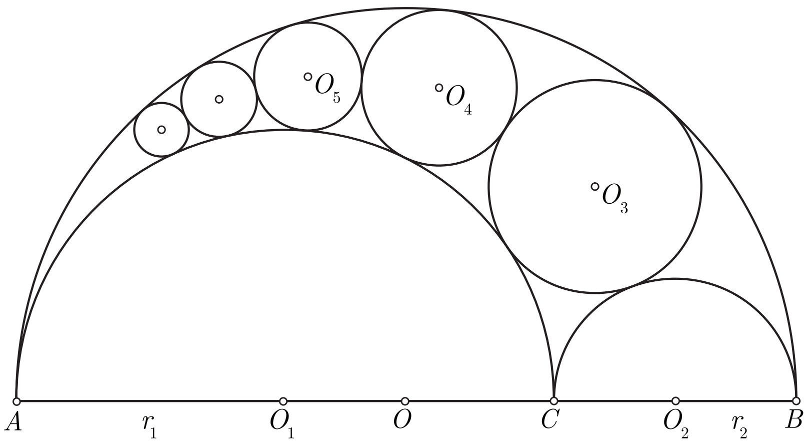

Fig. 3.

Case 3. (Fig. 4):

$$r_1 + r_2 = r_0, \qquad \frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2},$$

$$\sqrt{\frac{1}{r_1 r_3} - \frac{1}{r_3 r_0} - \frac{1}{r_0 r_1}} = \sqrt{\frac{1}{r_1}\left(\frac{1}{r_1} - \frac{1}{r_0}\right)},$$

From the same recurrence relation as in Case 2 and from formula (2′) we get formula for radius of n-th left circle

$$\frac{1}{r_n} = \left((n-3)\sqrt{\frac{1}{r_1} - \frac{1}{r_0}} + \sqrt{\frac{1}{r_1}}\right)^2 + \frac{1}{r_2}.$$

Similar is the formula for radius of n-th right circle

$$\frac{1}{r_n} = \left((n-3)\sqrt{\frac{1}{r_2} - \frac{1}{r_0}} + \sqrt{\frac{1}{r_2}}\right)^2 + \frac{1}{r_1}.$$

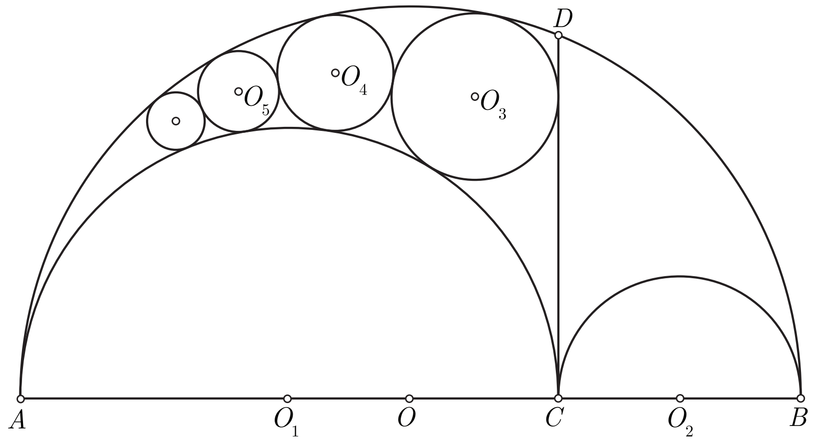

Fig. 4.

Case 4. (Fig. 5):

$$r_1 + r_2 = r_0, \qquad \frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2},$$

$$\sqrt{\frac{1}{4r_1r_2} + \frac{1}{2}\left(\frac{1}{r_1r_3} + \frac{1}{r_2r_3}\right)} = \frac{1}{2}\sqrt{\left(\frac{2}{r_1} + \frac{1}{r_2}\right)\left(\frac{2}{r_2} + \frac{1}{r_1}\right)}.$$

Formula for radius r_n is given by formula (2) if we take $r_1 \to 2r_1$, $r_2 \to 2r_2$.

$$\frac{1}{r_n} = \frac{(n-3)^2+2}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + (n-3)\sqrt{\left(\frac{2}{r_1} + \frac{1}{r_2}\right)\left(\frac{2}{r_2} + \frac{1}{r_1}\right)}.$$

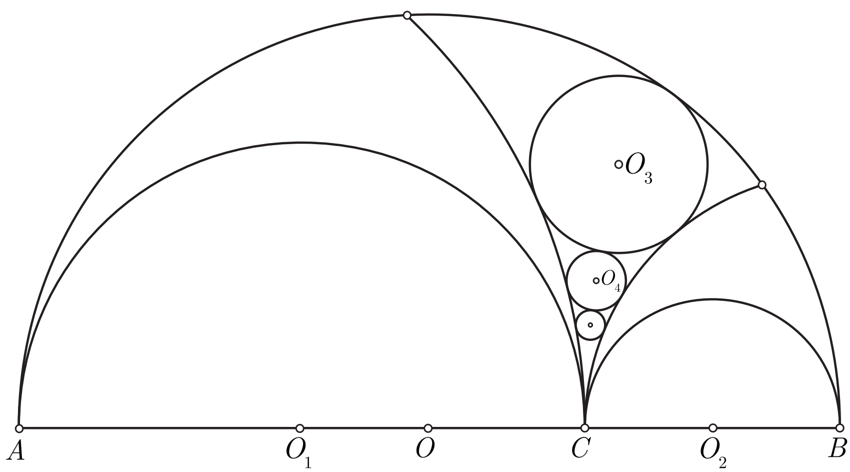

Fig. 5.

Case 5. (Fig. 6):

$$r_1 + r_2 = r_0 \qquad \frac{1}{r_3} = \frac{1}{4}\left(\frac{1}{r_1} - \frac{1}{r_0}\right) + \frac{1}{r_2},$$

$$\sqrt{\frac{1}{r_1 r_3} - \frac{1}{r_3 r_0} - \frac{1}{r_0 r_1}} = \frac{1}{2}\left(\frac{1}{r_1} - \frac{1}{r_0}\right).$$

From the same recurrence relation as in Case 2 and from formula (2′) we get

$$\frac{1}{r_n} = \left(n - \frac{5}{2}\right)^2 \left(\frac{1}{r_1} - \frac{1}{r_0}\right) + \frac{1}{r_2}.$$

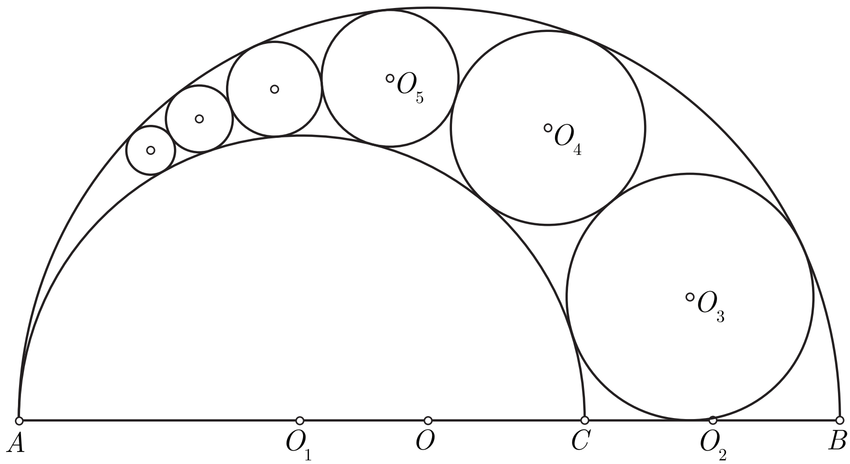

Fig. 6.

Case 6. (Fig. 7):

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}, \qquad \sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\sqrt{r_1 r_2}}.$$

Applying of formula (2) yields to

$$\frac{1}{r_n} = (n-2)^2 \left(\frac{1}{r_1} + \frac{1}{r_2}\right) + 2(n-2)\frac{1}{\sqrt{r_1 r_2}}.$$

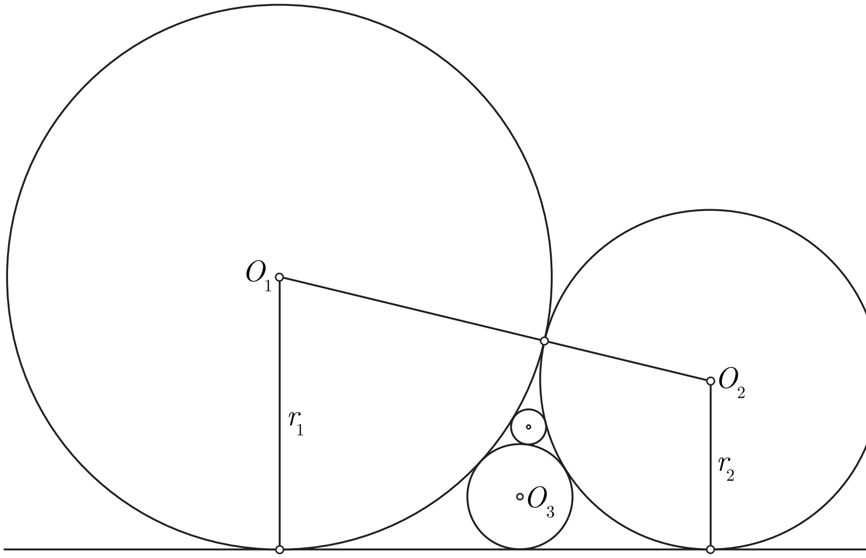

Fig. 7.

Theorem 2 (Pappus). *If the center of n-th circle with radius r_n inscribed in arbelos (see Fig. 8) is on the distance d_n of the base of arbelos then*

$$d_n = 2n \cdot r_n.$$

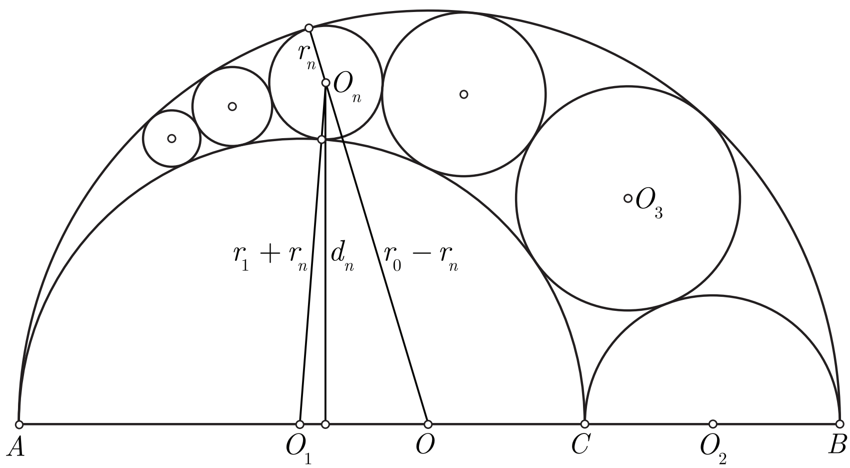

Fig. 8.

Proof. n-th circle inscribed in arbelos in our notation has radius r_{n+2} and distance d_{n+2} from the base of arbelos.Semiperimeter of triangle O_1OO_n is equal to r_0. From Archimedes-Heron formula for area of triangle, because of $r_1 + r_2 = r_0$, we have

$$2\sqrt{r_0 r_1 r_n (r_2 - r_n)} = r_2 \cdot d_n \quad \Rightarrow$$

$$\begin{aligned} d_n =& 2r_n \sqrt{\frac{r_0 r_1}{r_2}\left(\frac{1}{r_n} - \frac{1}{r_2}\right)} = \\ =& 2r_n \sqrt{\frac{r_0 r_1}{r_2}(n-2)^2 \frac{r_2}{r_0 r_1}} = 2(n-2)r_n. \end{aligned}$$

□

In the proof is used the formula for radius of n-th left circle in arbelos.

Theorem 3. *If the circles K_n, K_{n+1}, K_{n+2} and K_{n+3} are four consecutive circles from our sequence then for theirs radii we have*

$$\frac{1}{r_n} - 3\frac{1}{r_{n+1}} + 3\frac{1}{r_{n+2}} - \frac{1}{r_{n+3}} = 0.$$

Proof. Radii of these circles are given by unique formula

$$\frac{1}{r_k} = (k-3)^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{1}{r_3} + 2(k-3)\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}},$$
$$(k = n, n+1, n+2, n+3).$$

The desired formula follows after elimination of r_1, r_2 and r_3 from upper four formulas. □

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